ON FIXED POINT SETS OF HOMEOMORPHISMS OF THE n-BALL

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ABSTRACT

Conditions are investigated under which a subset A can be the fixed point set of a homeomorphism of B^n. If either $A \cap \partial B^n \neq \emptyset$ and n arbitrary or $A \cap \partial B^n = \emptyset$ and n even it is necessary and sufficient that A is non-empty and closed. If $A \cap \partial B^n = \emptyset$ and n odd, conditions which are either necessary or sufficient (but not both) are given.

1. **Introduction.** The purpose of this note is to find conditions under which a — by necessity non-empty and closed — subset A can be the fixed point set under a homeomorphism of the n-ball B^n. The answer depends on whether A contains a point of the boundary ∂B^n of B^n or not. If $A \cap \partial B^n \neq \emptyset$, then no further conditions are needed (Theorem 1). This is still true if $A \cap \partial B^n = \emptyset$ and n is even (Theorem 2), but not if n is odd. A necessary condition for this case is given in Theorem 3, a sufficient one in Theorem 6. A condition which is both necessary and sufficient still has to be found.—The case $n = 1$ is different and stated in Theorem 7.

The question as to which subsets can be fixed point sets under a homeomorphism of B^n was investigated by Robbins [2], who proved Theorem 2 for $n = 2$ and sketched a proof for the general case. He also proved a consequence of Theorem 3 and stated Theorem 1 for $n = 2$ but with an incorrect proof (the function f he constructs need not satisfy $f(B^2) \subseteq B^2$).

2. **The case** $A \cap \partial B^n \neq \emptyset$.

THEOREM 1. *If A is a non-empty closed subset of B^n with $A \cap \partial B^n \neq \emptyset$, then there exists for all $n > 1$ a homeomorphism of B^n with A as fixed point set.*

Proof. Let

$$I^n = \{x \in E^n \mid 0 \leqq x_i \leqq 1, \ i = 1, 2, \cdots, n\}$$

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be an n-cube in Euclidean space E^n, and let J^{n-1} be the union of all the $(n-1)$-faces of I^n apart from the one for which $x_n = 0$. We identify J^{n-1} to a point r_0 and denote the projection onto the quotient space by $p: I^n, J^{n-1} \to I^n/J^{n-1}, r_0$. Take a point s_0 of $A \cap \partial B^n$ and choose a homeomorphism $h: I^n/J^{n-1}, r_0 \to B^n, s_0$. Define $A' \subset I^n$ by $A' = (h \circ p)^{-1}(A)$; note that then $J^{n-1} \subset A'$.

We now construct a homeomorphism f' of I^n with fixed point set A'. Consider the map f' given by

$$f'(x_1, x_2, \cdots, x_n) = (x_1 + \tfrac{1}{2} d(x, A'), x_2, \cdots, x_n)$$

for all $x = (x_1, x_2, \cdots, x_n) \in I^n$, where $d(x, A')$ denotes the Euclidean distance of x from A'. As $(1, x_2, \cdots, x_n) \in A'$ we have $d(x, A') \leqq 1 - x_1$ and hence $x_1 + \frac{1}{2} d(x, A') \leqq 1$, so that $f'(I^n) \subseteq I^n$. The fixed point set of f' is clearly A'. It remains to be shown that f' is a homeomorphism.

Assume that f' is not injective so that there exist points $y, z \in I^n$ with $f'(y) = f'(z)$ but $y \neq z$. Then $y_1 \neq z_1$, $y_i = z_i$ for $i = 2, 3, \cdots, n$, and we can assume that $y_1 > z_1$. As $f'(y) = f'(z)$ we have

$$y_1 + \tfrac{1}{2} d(y, A') = z_1 + \tfrac{1}{2} d(z, A').$$

Using $d(y, z) = y_1 - z_1$ we obtain

$$2d(y, z) + d(y, A') = d(z, A').$$

But it is necessary that

$$d(z, y) + d(y, A') \geqq d(z, A'),$$

which contradicts the previous formula. Therefore f' is injective.

In order to see that f' is surjective it is sufficient to see that it is surjective on each interval I of I^n parallel to the x_1 – axis. But this follows from the fact that $f | I: I \to I$ is a continuous function which leaves the two end points of I fixed. Hence f' is a homeomorphism of I^n with fixed point set A'.

Now define $f: B^n \to B^n$ by

$$f = h \circ p \circ f' \circ p^{-1} \circ h^{-1}.$$

As p is a projection and $f' \circ p^{-1}$ is single-valued it follows that $f' \circ p^{-1}$ is continuous, so that f is a map. From its construction we see that f is a homeomorphism of B^n with fixed point set A as required.

3. **The case** $A \cap \partial B^n = \emptyset$.

THEOREM 2. (*Robbins*). *If A is a non-empty closed subset of B^{2n} with $A \cap \partial B^{2n} = \emptyset$, then there exists a homeomorphism of B^{2n} with A as fixed point set.*

Proof. Without loss of generality we can assume that B^{2n} is given by

$$B^{2n} = \left\{ x = (x_1, x_2, \cdots, x_{2n}) \in E^{2n} \,\middle|\, \sum_{k=1}^{2n} x_k^2 \leqq 1 \right\}$$

and that $0 = (0, 0, \cdots, 0) \in A$. Define a map $f: B^{2n} \to B^{2n}$ by

$$f(x_1, x_2, \cdots, x_{2n}) = (x_1', x_2', \cdots, x_{2n}'),$$

where

$$\begin{aligned} x_{2i-1}' &= x_{2i-1}\cos\psi + x_{2i}\sin\psi \\ x_{2i}' &= -x_{2i-1}\sin\psi + x_{2i}\cos\psi \end{aligned} \qquad (i = 1, 2, \cdots, n)$$

with

$$\psi = d(x, A).$$

Then A is the fixed point set of f. We show that f is a homeomorphism.

Let y, z be two points of B^{2n} such that $f(y) = f(z)$; we can assume that $d(y, A) \geqq d(z, A)$. Put

$$\theta = d(y, A) - d(z, A).$$

As $0 \leqq d(x, A) \leqq 1$ for all $x \in B^{2n}$, we have $0 \leqq \theta \leqq 1$. Now consider the projections $p_i(y)$, $p_i(z)$, and $p_i f(y)$ of y, z, and $f(y) = f(z)$ into the (x_{2i-1}, x_{2i})-plane. $p_i f(y)$ is obtained from $p_i(y)$ and $p_i(z)$ by rotation on a circle with centre 0 and raduis $r_i = \sqrt{(y_{2i-1}^2 + y_{2i}^2)}$ through angles $d(y, A)$ and $d(z, A)$ respectively. Hence

$$d(p_i(y), p_i(z)) = 2r_i \sin\theta/2.$$

But

$$d(y, z) = \sqrt{\left[\sum_1^n d^2(p_i(y), p_i(z))\right]},$$

therefore

$$d(y, z) = 2\sqrt{\left(\sum_1^n r_i^2\right)} \sin\theta/2.$$

As $\sum_1^n r_i{}^2 = d^2(y, 0) \leqq 1$, we obtain

$$d(y,z) \leqq 2\sin\theta/2.$$

From

$$d(y,z) \geqq d(y,A) - d(z,A) = \theta$$

it follows that

$$\theta \leqq 2\sin\theta/2,$$

which is only possible if $\theta = 0$, i.e. if $d(y,A) = d(z,A)$. Therefore we have $p_i(y) = p_i(z)$ for $i = 1,2,\cdots,n$, and hence $y = z$. So f is injective.

As $|f(x)| = |x|$ for all $x \in B^{2n}$ it follows that f is surjective if it is surjective on every $(2n-1)$-sphere $|x| = c$, $0 < c \leqq 1$. The homotopy

$$f_t(x_1,x_2,\cdots,x_{2n}) = (x_1',x_2',\cdots,x_{2n}')$$

defined by

$$\begin{aligned} x_{2i-1}' &= x_{2i-1}\cos t\psi + x_{2i}\sin t\psi \\ & \qquad\qquad\qquad\qquad (i = 1,2,\cdots,n; 0 \leqq t \leqq 1) \\ x_{2i}' &= -x_{2i-1}\sin t\psi + x_{2i}\cos t\psi \end{aligned}$$

shows that on every such $(2n-1)$-sphere the restriction of the map $f = f_1$ is homotopic to the identity map f_0. Therefore it is essential and must be surjective. This completes the proof of Theorem 2.

We now derive some necessary conditions on A for balls of odd dimension.

THEOREM 3. *If the subset A of B^{2n+1} is the fixed point set of a homeomorphism of B^{2n+1} and if $A \cap \partial B^{2n+1} = \varnothing$, then A contains no $2n$-sphere.*

Proof. (Robbins). Let A contain a $2n$-sphere; we can assume that it is the set $\{x \in E^{2n+1} \mid |x| = \frac{1}{2}\}$ inside $B^{2n+1} = \{x \in {}^{2n+1} \mid |x| \leqq 1\}$. Define a homotopy $f_t\colon \partial B^{2n+1} \to \partial B^{2n+1}$ by

$$f_t(x) = f\left(\frac{t+1}{2}x\right) \Big/ \left| f\left(\frac{t+1}{2}x\right)\right|, \ (x \in \partial B^{2n+1}, 0 \leqq t \leqq 1).$$

Then $f_0(x) = x$ and $f_1(x) = f(x)$, so that the restriction of f to ∂B^{2n+1} is a map of an even-dimensional sphere which is homotopic to the identity and fixed point free. But this is impossible.

COROLLARY 4. *If the subset A of B^{2n+1} is the fixed point set of a homeomorphism of B^{2n+1} and if $A \cap \partial B^{2n+1} = \varnothing$, then the interior of A is empty.*

Proof. Any point of the interior of A would have a neighbourhood in A which contains a $2n$-sphere.

COROLLARY 5. *If the subset A of B^{2n+1} is the fixed point set of a homeomorphism of B^{2n+1} and if $A \cap \partial B^{2n+1} = \varnothing$, then $\dim A < 2n+1$.*

Proof. If $\dim A = 2n+1$ then A contains a non-empty subset which is open in B^{2n+1} (see [1, p. 44]). Hence the interior of A would not be empty.

THEOREM 6. *Let $n \geqq 1$. If the non-empty closed subset A of B^{2n+1} is contained in a hyperplane, then there exists a homeomorphism of B^{2n+1} with A a fixed point set.*

Proof. It follows from Theorem 1 that we only have to prove the case where $A \cap \partial B^{2n+1} = \varnothing$. Let then $B^{2n+1} = \{x \in E^{2n+1} \mid |x| \leqq 1\}$ and $B^{2n} = \{x \in B^{2n+1} \mid x_{2n+1} = 0\}$. We can assume that $A \subset B^{2n}$ and $0 \in A$. Construct a homeomorphism f_{2n} of B^{2n} according to Theorem 2 and extend it over B^{2n+1} by defining

$$f(x_1, x_2, \cdots, x_{2n+1}) = (f_{2n}(x_1, x_2, \cdots, x_{2n}), -x_{2n+1}).$$

Then the homeomorphism f satisfies Theorem 6.

4. **The case** $n = 1$. The case $n = 1$ follows easily from the fact that a homeomorphism of B^1 is a strictly monotonic function. We state the result.

THEOREM 7. *The subset A of B^1 is the fixed point set of a homeomorphism of B^1 if and only if one of the following two conditions is true:*

(i) *A consists of a single point in the interior of B^1,*

(ii) *A is non-empty, closed, and $\partial B^1 \subseteq A$.*

REFERENCES

1. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton (1948).

2. H. Robbins, *Some complements to Brouwer's fixed point theorem*, Israel J. Math. **5** (1967), 225–226.

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